Pricing Kernel Monotonicity and Conditional Information

Pricing Kernel

- 反映了投資者對財富的邊際效用

$$p_t = E_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1}\right]. \quad p_t = E_t(m_{t+1} x_{t+1}).$$

"classical" nonparametric SDF estimator. Existing research has found that it is typically a decreasing function of the market return over much of its range, but it is also often increasing over part of its range. Important improvements in the classical method over time have not changed this result.[1]

An investor's first-order conditions give the basic consumption-based model,

$$p_t = E_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1}\right].$$

Let us find the value at time t of a *payoff* x_{t+1}. If you buy a stock today, the payoff next period is the stock price plus dividend, $x_{t+1} = p_{t+1} + d_{t+1}$. x_{t+1} is a random variable: an investor does not know exactly how much he will get from his investment, but he can assess the probability of various possible outcomes. Do not confuse the *payoff* x_{t+1} with the *profit* or *return*; x_{t+1} is the value of the investment at time $t + 1$, without subtracting or dividing by the cost of the investment.

Pricing Kernel

- 在資產定價中，**pricing kernel**為轉換真實機率測度到風險中立機率測度的工具

$$m_t(x_{t+s}) = e^{-rs} \frac{d\mathbb{F}^Q(x_{t+s}|\mathcal{F}_t)}{d\mathbb{F}^P(x_{t+s}|\mathcal{F}_t)},$$

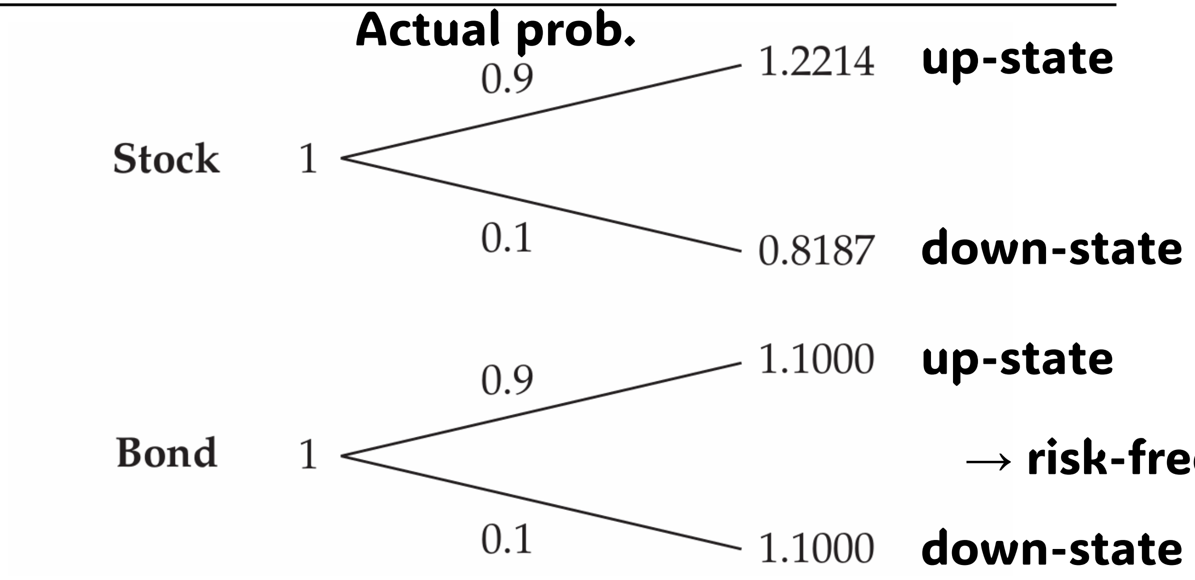
- **If FQ and FP are measures induced by the risk-neutral and physical cumulative distribution functions, the SDF can be expressed as a change of measure between two conditional probability measures where each probability is conditional on the same information set,Ft.**

Pricing Kernel

$$m_t(x_{t+s}) = e^{-rs} \frac{d\mathbb{F}^Q(x_{t+s}|\mathcal{F}_t)}{d\mathbb{F}^P(x_{t+s}|\mathcal{F}_t)},$$

- 當財富上升，**risk-aversion pricing kernels**直覺應下降，但研究表明並非如此

Figure 2. Simple One-Period, Two-State Economy



$(S_t - K)^+$

0.2214

0

→ risk-free rate = 10%

此為定價錯誤例子其實應使用risk-neutral prob.

Call (K = 1)
=(0.2214×0.9+0×0.1)/0.1
=0.1993/0.1
=0.1811

But, the true price is only 0.1406

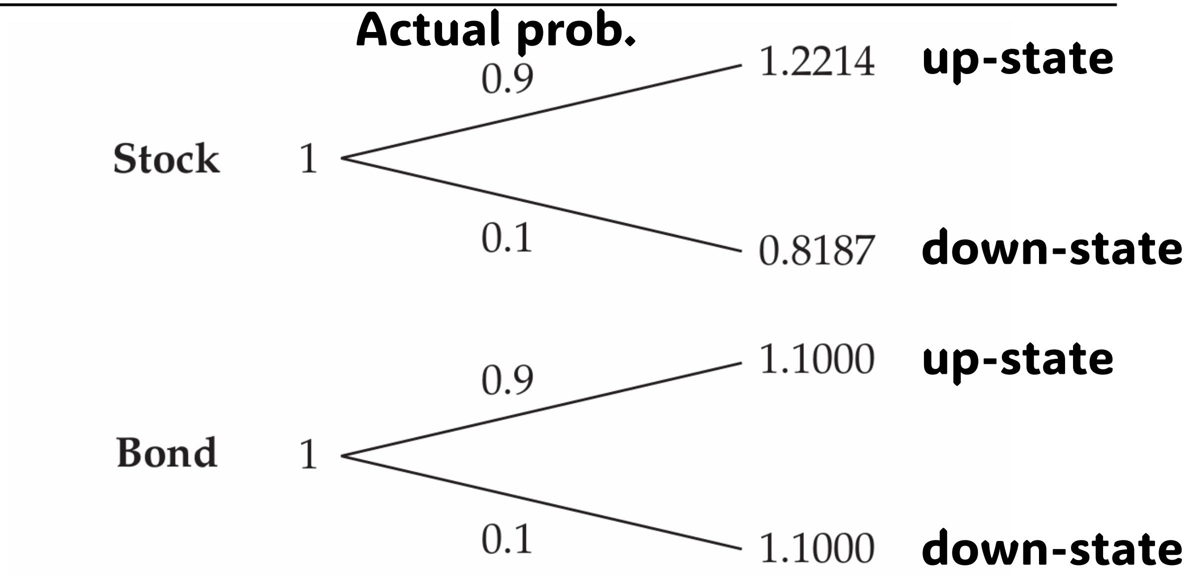
Pricing Kernel

$$m_t(x_{t+s}) = e^{-rs}\frac{d\mathbb{F}^Q(x_{t+s}|\mathcal{F}_t)}{d\mathbb{F}^P(x_{t+s}|\mathcal{F}_t)},$$

- 當財富上升，**risk-aversion pricing kernels**直覺應下降，但研究表明並非如此

The reason for the difference is that investors receive the payoff from the call in the up state when they are already wealthy. In that state, they have less appreciation for additional cash flows and will accordingly pay less for the call. Therefore, to work out the true price of the call, we must turn to the concept of STATE PRICES. The state price is what an investor is willing to pay for the certain payment of 1.00 in the certain state.

Figure 2. Simple One-Period, Two-State Economy



State price

π_u

π_d

π_u

π_d

$$\rightarrow 1 = \pi_u 1.2214 + \pi_d 0.8187$$

$$\rightarrow 1 = \pi_u 1.1 + \pi_d 1.1$$

$$\pi_u = 0.6350, \pi_d = 0.2741$$

State price

State price 代表市場「現在」願意支付多少錢，以獲得特定未來狀態下的 1 塊錢報酬。(已折現)

例子 1：投保概念

假設有兩種可能的市場狀態：

1. **股市大漲（好狀況）**，發生機率 80%。
2. **股市大跌（壞狀況）**，發生機率 20%。

你想要在「壞狀況」發生時確保自己可以拿到 1 塊錢，所以你今天去買一個保險合約。

這時，**state price 就是你今天需要支付的價格**，來確保如果「壞狀況」發生時，你能拿到 1 塊錢。例如：

- 如果市場認為壞狀況很可能發生（市場恐慌），你今天可能要花 **0.6 元** 來確保未來能拿到 1 塊錢。
- 如果市場認為壞狀況不太可能發生（市場樂觀），你今天可能只需要花 **0.1 元**。

這說明 **state price 是現在的價格，但它代表的是未來某個特定狀態的價值**。

6. State Price 與風險中立機率（Risk-Neutral Probability）的關係

- **風險中立機率（Risk-Neutral Probability, $q(S_T)$）是用來計算衍生性金融商品價格的機率分布**，而 State price 與它有以下關係：

$$\text{State Price} = e^{-r(T-t)}q(S_T)$$

- 這表示：
  - **State price = 貼現後的風險中立機率**。
  - 這個關係被用來計算選擇權價格，例如在 Black-Scholes 模型中，選擇權價格來自不同狀態的 State price 加總。

✅ **State price 與未來狀態的機率正相關，但風險趨避可能改變這種關係。**

✅ **當某個狀態的機率變高，state price 可能上升，但如果市場不害怕該狀態，state price 仍可能較低。**

✅ **當某個狀態的機率變低，state price 可能下降，但如果市場特別害怕該狀態，state price 可能仍維持高位。**

Pricing Kernel

$$m_t(x_{t+s}) = e^{-rs}\frac{d\mathbb{F}^Q(x_{t+s}|\mathcal{F}_t)}{d\mathbb{F}^P(x_{t+s}|\mathcal{F}_t)},$$

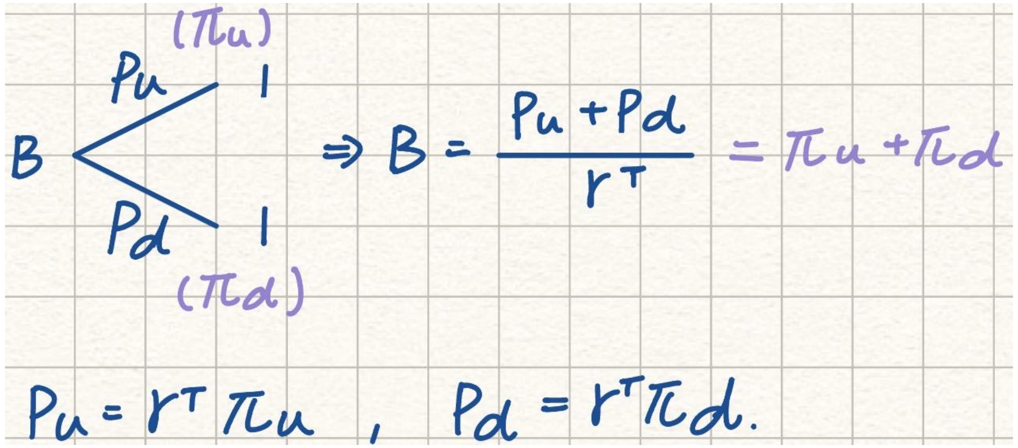
$\pi_u = 0.6350, \pi_d = 0.2741$

- 當財富上升，**risk-aversion pricing kernels**直覺應下降，但研究表明並非如此

- **The sum of the state prices has to be equal to the price of a bond, which pays 1.00 in each state (1/1.1 = 0.9091).** $1 = \pi_u 1.1 + \pi_d 1.1 \rightarrow 0.9091 = \pi_u + \pi_d$
- **A convenient step to get risk-neutral probabilities is multiplying the state prices by the inverse of the price of this unit bond.**
- **inverse of the price = 1/0.9091 = 1.1 = risk-free rate** r^T

$$P_i = r^T \pi_i,$$

$$P_u = 0.6350 \times 1.1 = 0.6985$$



$$P_d = 0.2741 \times 1.1 = 0.3015.$$

Call price

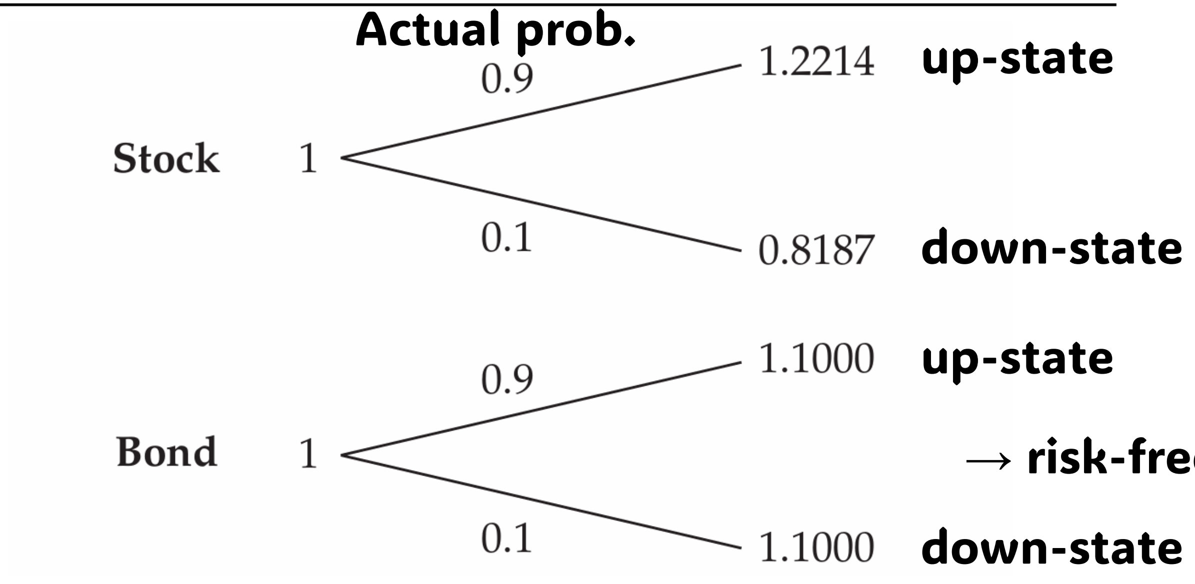
$$= \frac{\sum_i P_i\, payoff_i}{r^T}$$

Pricing Kernel

$$m_t(x_{t+s}) = e^{-rs} \frac{d\mathbb{F}^Q(x_{t+s}|\mathcal{F}_t)}{d\mathbb{F}^P(x_{t+s}|\mathcal{F}_t)},$$

- 當財富上升，**risk-aversion pricing kernels**直覺應下降，但研究表明並非如此

Figure 2. Simple One-Period, Two-State Economy



→ **risk-free rate = 10%**

$(S_t - K)^+$

0.2214

0

risk-neutral probabilities

$P_u = 0.6985$

$P_d = 0.3015$

Call (K = 1)

$= \frac{(0.6985 \times 0.2214 + 0.3015 \times 0)}{1.1}$

$= \mathbf{0.1406}$

$\pi_u = 0.6350, \pi_d = 0.2741$

Pricing Kernel

$$m_t(x_{t+s}) = e^{-rs} \frac{d\mathbb{F}^Q(x_{t+s}|\mathcal{F}_t)}{d\mathbb{F}^P(x_{t+s}|\mathcal{F}_t)},$$

- 當財富上升，**risk-aversion pricing kernels**直覺應下降，但研究表明並非如此

The ratio of state prices to actual probabilities is called the PRICING KERNEL(m) or STOCHASTIC DISCOUNT FACTOR.
Define m = π/Q, where Q is the actual probability.

$$P_i = r^T \pi_i,$$

m = $e^{-rT}\frac{P}{Q}$

P = $e^{rT}\pi \rightarrow e^{-rT}\mathrm{P} = \pi$ $\rightarrow m = \frac{\pi}{Q}$

Call price

$$= \frac{\sum_i P_i \, payoff_i}{r^T}$$

$$= \sum_i \frac{Q_i}{Q_i} \pi_i \;\; payoff_i$$

$$= \sum_i m_i Q_i \;\; payoff_i$$

$$m_i = \frac{\pi_i}{actual\ prob_i} = \frac{\frac{risk\ neutral\ prob_i}{r^T}}{actual\ prob_i} = r^{-T} \frac{risk\ neutral\ prob_i}{actual\ prob_i}$$

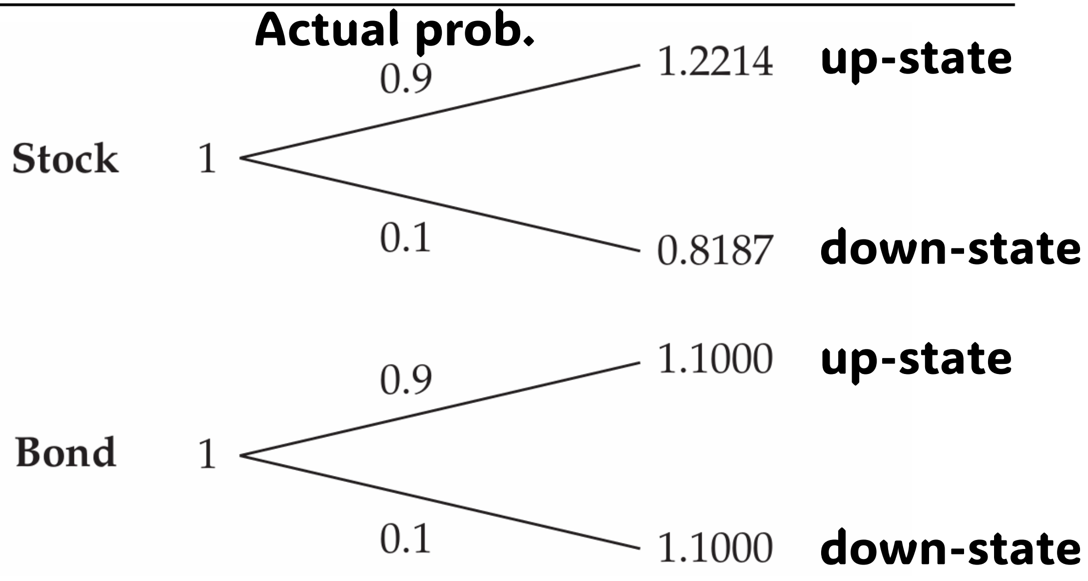
Pricing Kernel

$$m_t(x_{t+s}) = e^{-rs} \frac{d\mathbb{F}^Q(x_{t+s}|\mathcal{F}_t)}{d\mathbb{F}^P(x_{t+s}|\mathcal{F}_t)},$$

- 當財富上升，**risk-aversion pricing kernels**直覺應下降，但研究表明並非如此

The ratio of state prices to actual probabilities is called the PRICING KERNEL(m) or STOCHASTIC DISCOUNT FACTOR. Define m = π/Q, where Q is the actual probability.

Figure 2. Simple One-Period, Two-State Economy



State price

$\pi_u = 0.6350$

$\pi_d = 0.2741$

Pricing kernel

$m_u = \frac{0.6350}{0.9}$ = **0.7056**

$m_d = \frac{0.2741}{0.1}$ = **2.7410**

Pricing Kernel

$$m_t(x_{t+s}) = e^{-rs} \frac{d\mathbb{F}^Q(x_{t+s}|\mathcal{F}_t)}{d\mathbb{F}^P(x_{t+s}|\mathcal{F}_t)},$$

- 當財富上升，**risk-aversion pricing kernels**直覺應下降，但研究表明並非如此

The ratio of state prices to actual probabilities is called the PRICING KERNEL(m) or STOCHASTIC DISCOUNT FACTOR. Define m = π/Q, where Q is the actual probability.

Pricing kernel

$$m_u = \frac{0.6350}{0.9} = \mathbf{0.7056}$$

$$m_d = \frac{0.2741}{0.1} = \mathbf{2.7410}$$

- **When the investor is poor, he or she treats 1.00 received as if it happened with about three times the likelihood that it really did happen. When the investor is rich, he or she treats 1.00 received as if it happened with only about three-quarters of the likelihood that it really did.**
- **Such an investor is risk averse because he or she does not like exposure to the down state at all and is willing to pay to avoid it.**
- **For a risk-averse investor, the pricing kernel is decreasing in wealth, as is the case here**

Theoretical Pricing Kernel 下降 ⇒ 投資人較不重視此時的 1 單位報酬
Theoretical Pricing Kernel 上升 ⇒ 投資人較重視此時的 1 單位報酬

Consistency

- **Most estimates are inconsistent for the true pricing kernel because they compare a forward-looking, conditional risk-neutral density estimated with option prices to a backward-looking, essentially unconditional physical density estimated with historical returns.**

Researchers should take care to estimate the densities in a conditional, forward-looking manner. For estimation of the numerator, most researchers rely on the result of Breeden and Litzenberger (1978), that $\frac{d\mathbb{F}^Q}{dK} = e^{rT}\frac{\partial^2 C}{\partial K^2}$, where C represents the option price, K represents strike prices and $\frac{d\mathbb{F}^Q}{dK}$ represents the risk-neutral density over possible realizations of the underlying. Since we typically observe option prices with a number of strike prices K, we are able to estimate the derivative $\frac{d\mathbb{F}^Q}{dK}$ over a collection of points K. Various techniques for estimating or interpolating values of the density between observed strike prices have been proposed in the literature. This gives an estimate of the risk-neutral density that is forward-looking and conditional on all information investors use to set prices.

Consistency

Assumption 1. The true SDF is proportional to the ratio of two densities, both of which are conditional on all available information available at t: $m_t(x_{t+s}) = e^{-rs} \frac{d\mathbb{F}^Q(x_{t+s}|\mathcal{F}_t)}{d\mathbb{F}^P(x_{t+s}|\mathcal{F}_t)}$.

Assumption 2. Econometricians only observe a subset of all information, $\mathcal{S}_t \subset \mathcal{F}_t$.

Assumption 3. Econometricians use estimators that are pointwise consistent for the risk-neutral density conditional on $\mathcal{F}_t$ and the physical density conditional on $\mathcal{S}_t$, denoted $d\mathbb{F}^{Q*}(x_{t+s}|\mathcal{F}_t)$ and $d\mathbb{F}^{P*}(x_{t+s}|\mathcal{S}_t)$ respectively.

Consistency

Proposition 1. Given Assumptions 1–3, the classical estimate $m_t^*(x_{t+s}) = e^{-rs}\frac{d\mathbb{F}^{Q*}(x_{t+s}|\mathcal{F}_t)}{d\mathbb{F}^{P*}(x_{t+s}|\mathcal{S}_t)}$ is pointwise consistent for the true SDF, $m_t(x_{t+s})$, if and only if $d\mathbb{F}^P(x_{t+s}|\mathcal{F}_t) = d\mathbb{F}^P(x_{t+s}|\mathcal{S}_t)$.

The proof of this proposition is very simple. Since we have assumed that we have two pointwise consistent estimators, the classical estimate converges pointwise to

$$m_t^*(x_{t+s}) \underset{\lim}{\longrightarrow} e^{-rs}\frac{d\mathbb{F}^Q(x_{t+s}|\mathcal{F}_t)}{d\mathbb{F}^P(x_{t+s}|\mathcal{S}_t)} = m_t(x_{t+s})\left[\frac{d\mathbb{F}^P(x_{t+s}|\mathcal{F}_t)}{d\mathbb{F}^P(x_{t+s}|\mathcal{S}_t)}\right]. \tag{3}$$

$$m_t^*(x_{t+s}) = e^{-rs}\frac{dF^{Q*}(x_{t+s}|F_t)}{dF^{P*}(x_{t+s}|S_t)} \xrightarrow[\lim]{\text{Ass. 3}} e^{-rs}\frac{dF^{Q}(x_{t+s}|F_t)}{dF^{P}(x_{t+s}|S_t)}$$

$$m_t(x_{t+s}) = e^{-rs}\frac{dF^{Q}(x_{t+s}|F_t)}{dF^{P}(x_{t+s}|F_t)} \Rightarrow m_t^* = m_t\left[\frac{dF^P(x_{t+s}|F_t)}{dF^P(x_{t+s}|S_t)}\right]$$

Consistency

Proposition 1. Given Assumptions 1–3, the classical estimate $m_t^*(x_{t+s}) = e^{-rs}\frac{d\mathbb{F}^{Q*}(x_{t+s}|\mathcal{F}_t)}{d\mathbb{F}^{P*}(x_{t+s}|\mathcal{S}_t)}$ is pointwise consistent for the true SDF, $m_t(x_{t+s})$, if and only if $d\mathbb{F}^P(x_{t+s}|\mathcal{F}_t) = d\mathbb{F}^P(x_{t+s}|\mathcal{S}_t)$.

The proof of this proposition is very simple. Since we have assumed that we have two pointwise consistent estimators, the classical estimate converges pointwise to

$$m_t^*(x_{t+s}) \underset{\lim}{\longrightarrow} e^{-rs}\frac{d\mathbb{F}^Q(x_{t+s}|\mathcal{F}_t)}{d\mathbb{F}^P(x_{t+s}|\mathcal{S}_t)} = m_t(x_{t+s})\left[\frac{d\mathbb{F}^P(x_{t+s}|\mathcal{F}_t)}{d\mathbb{F}^P(x_{t+s}|\mathcal{S}_t)}\right]. \quad (3)$$

Given that many of the nonmonotonicity papers use only past index prices to estimate the physical density, this assumption seems unlikely to hold in practice.

CDI method (Conditional density integration)

- **risk-neutral density**

1. **fit a fourth-degree spline to implied volatilities associated with each observed strike price.**

   → create a continuous curve in the implied volatility space.

2. **convert the implied volatility curve back to the price space**

3. $\frac{d\mathbb{F}^Q}{dK} = e^{rT}\ \frac{\partial^2 C}{\partial K^2}$ **where $\mathbb{F}^{\mathrm{Q}}$ represents the risk-neutral CDF and $\frac{d\mathbb{F}^{\mathrm{Q}}}{dK}$ represents the density over prices, K.**

CDI method (Conditional density integration)

For any continuous random variable X with CDF $\mathbb{F}$, the random variable defined by $\mathbb{F}(X)$ is uniformly distributed on the interval $[0,1]$,

$$\mathbb{F}(X) \sim U[0,1]. \tag{4}$$

We let $\mathbb{F}^P(x_{t+s}|\mathcal{F}_t)$ be the unobserved probability measure representing investors' aggregate beliefs about returns on the S&P 500 under the physical measure between time t and $t+s$ and let returns over the subsequent period be given by x_{t+s}. Now it follows from Equation 4, that

$= F^P(X_{t+s} | \mathcal{F}_t) - F^P(-\infty | \mathcal{F}_t)$

$=$ CDF of X_{t+s} under P measure

$$\int_{-\infty}^{x_{t+s}} d\mathbb{F}^P(x_{t+s}|\mathcal{F}_t) \sim U[0,1]. \tag{5}$$

$$\int_{-\infty}^{x_{t+s}} d\mathbb{F}^P(x_{t+s}|\mathcal{F}_t) = \int_{-\infty}^{x_{t+s}} \underbrace{\boxed{\frac{d\mathbb{F}^P(x_{t+s}|\mathcal{F}_t)}{d\mathbb{F}^Q(x_{t+s}|\mathcal{F}_t)}}}_{g_{t,s}} d\mathbb{F}^Q(x_{t+s}|\mathcal{F}_t)$$

$$= \int_{-\infty}^{x_{t+s}} \left(\frac{d\mathbb{F}^Q(x_{t+s}|\mathcal{F}_t)}{d\mathbb{F}^P(x_{t+s}|\mathcal{F}_t)}\right)^{-1} d\mathbb{F}^Q(x_{t+s}|\mathcal{F}_t) \sim U[0,1]. \tag{6}$$

Spline

Spline（樣條），數學上，通常指一條「分段多項式曲線」（piecewise polynomial curve），各段在節點（knots）處滿足某種連續性條件（例如函數值、導數值連續）。最常見的是「三次樣條」（Cubic Spline），即每一段是三次多項式，並在節點上保持函數與一階、二階導數連續。

主要特性

- 分段性：將定義域切分成若干小區間，每段使用一個低次多項式表示。
- 平滑性：在相鄰區間的分界節點，要求函數值與導數（常至少到二階）連續，達到整條曲線平滑過渡。
- 插值或擬合：Spline 可以用來插值，即通過給定的「觀測點」精確匹配，也可以用來回歸/擬合（加入誤差項或懲罰項），以平滑方式近似一組散亂數據。

分段三次多項式的概念假設你有 n+1 個給定資料點 $\{(x0,y0),(x1,y1),\ldots,(xn,yn)\}$，想要找一條平滑曲線去「插值」經過這些點。「一般三次 Spline 插值」就是在每對相鄰節點 $[xi,\ xi+1]$上，分別用一段三次多項式$Si(x,x\in[xi,\ xi+1]$來描述。這樣總共會有 n 段，每段各有 4 個係數 $(ai,bi,ci,di$。整條插值曲線由這 n段連續拼起來。

Clamped Spline

Clamped Spline (固定端點樣條)，其主要特點是對「端點節點」（knot）做重複（multiplicity），使得在最左最右兩端，Spline 函數的值、以及一階、二階導數能以特定方式貼合「邊界條件」。
最常見的做法是對三次樣條（order=4），把左右端點各重複 4 次

主要特性

- 在 最左端 x = a 上，Clamped Spline 強制使曲線值 $S(a)$S(a) 恰好等於「觀測（或指定）的邊界值」。同時通過重複節點，也讓那條基底的導數或二階導數在 x=a 保持平滑連續。同理，在 最右端 x = b 上，也能將曲線值 $S(b)$S(b) 固定於某一數值。

- 這種「clamped」方式確保曲線在端點極為貼合給定的邊界要求，也能保持端點附近 “曲率有限” ，不會像自由曲線那樣邊界劇烈擺動

Natural Spline

Natural Spline（自然樣條）是一種在端點二階導數設為零的三次樣條插值方法。具體要求：在所有節點 xix i 處，Spline 會精確通過 (xi,yi)，且保持一階、二階導數連續。在最左端 $x=a$x=a 和最右端 $x=b$x=b 處，強制 $S''(a)=0$、$S''(b)=0$，也就是「邊界上無曲率，匯合到一個平直狀態」。

主要特性

- Natural Spline 不會硬性「釘住某條基底在端點取 1」，而是在端點讓「曲率 = 二階導數 = 0」。因此曲線在邊界附近往往會以「線性」或「次線性」方式延展，減少外推時的劇烈彎曲。

- 相比 Clamped Spline，Natural Spline 更像是在端點「自然放手」：曲線兩端不再被強制到某個值，而是根據「沒有曲率」的條件往外平滑延伸。

B-Spline

B-spline（Basis spline）是「樣條基底函數」的縮寫。B-spline 本身並不是一條具體的曲線，而是一組「分段多項式基底」函數。透過 B-spline basis，可以用線性組合的方式拼湊出任意高階樣條曲線。

特性

- 特性局部支撐（Local Support）：每個 B-spline basis $Bj(x)$只在某段非常有限的「支撐區間」（support interval）內非零，離開該區間即為 0。這使得當某個控制點或觀測點改變時，只會影響鄰近區域的形狀，曲線其他部分保持不變。
- 最低階多項式：常見的是「三次 B-spline」，也就是 order = 4（多項式階數 = 3）。它可在每個節點（knot）與鄰近節點之間以三次多項式連接。
- 參數化穩定：相比直接用高階全域多項式擬合，B-spline 具有更好的數值穩定性與控制性。

B-spline （Basis spline）一套「用基底函數去表示任意樣條曲線」的框架。
換句話說，你可以把所有的分段三次多項式插值，都寫成「若干條 B-spline 基底 $Bj(x)$ 」的線性組合：

$$S(x) \ = \ \sum_j c_j \, B_j(x),$$

其中每條 $Bj(x)$ 都是一個局部有非零支撐的三次多項式基底（它只在某一小段區間非零）。
集合這些基底，再搭配合適的係數 $\{cj\}$，就能拼出插值曲線或擬合曲線。

Basis function如何取值

Basis function會在指定節點區間中作用

假設有節點$t_1, t_2, \ldots\ldots, t_n$

有一0次Basis function在$[t_1, t_2]$中作用→ $\mathrm{B}^0_{[t_1,\ t_2]}(x) = \begin{cases} 1, t_1 \le x < t_2 \\ 0, otherwise \end{cases}$

n次Basis function在$[t_i, t_{i+n+1}]$中作用

$\mathrm{B}^n_{[t_i,\ t_{i+n+1}]}(x) = \frac{x-t_i}{t_{i+n}-t_i}\mathrm{B}^{n-1}_{[t_i,\ t_{i+n}]}(x) + \frac{t_{i+n+1}-x}{t_{i+n+1}-t_{i+1}}\mathrm{B}^{n-1}_{[t_{i+1}\ \ t_{i+n+1}]}(x)$ … 遞迴公式

以論文中程式邏輯及4base cubic B spline 為例：

給定區間[min,max]，要算出$B^{3,1}_{[min,max]}(x)$, $B^{3,2}_{[min,max]}(x)$, $B^{3,3}_{[min,max]}(x)$, $B^{3,4}_{[min,max]}(x)$

如何取knots？

需要的knots數(包含邊界) = base-2 = 2
會在[min,max]均勻取點
邊界節點重複4次，確保靠近邊界的點可控

以4base cubic B spline 為例: knots = $min, min, min, min, max, max, max, max = t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8$

後續以n代替min, m代替max

$$B^{3,1}_{[n,m]}(x)\text{在}[t_1, t_5]\text{上作用} = \begin{cases} 1, if\ x = n \\ 0 < \dfrac{(m-x)^3}{(m-n)^3} < 1, if\ n < x < m \\ 0, otherwise \end{cases}$$

$$B^{3,2}_{[n,m]}(x)\text{在}[t_2, t_6]\text{上作用} = \begin{cases} 0 < 3\dfrac{(x-n)(m-x)^2}{(m-n)^3} < 3, if\ n < x < m \\ 0, \quad otherwise \end{cases}$$

$$B^{3,3}_{[n,m]}(x)\text{在}[t_3, t_7]\text{上作用} = \begin{cases} 0 < 3\dfrac{(x-n)^2(m-x)}{(m-n)^3} < 3, if\ n < x < m \\ 0, \quad otherwise \end{cases}$$

$$B^{3,4}_{[n,m]}(x)\text{在}[t_4, t_8]\text{上作用} = \begin{cases} 1, if\ x = m \\ 0 < \dfrac{(x-n)^3}{(m-n)^3} < 1, if\ n < x < m \\ 0, otherwise \end{cases}$$

✅ 選項一：將內部區間切得更細

（也就是在中間加更多 knot）

➕ 優點（但有限）：

- 可以讓 spline 曲線在「中間區域」更靈活、精細；
- 適合捕捉區間中間的快速變化（像高峰、高斜率）。

➖ 缺點：

- 對「邊界端點」幫助不大 —— 因為 spline 函數的控制力仍然漸進接近0；
- 端點處的基底函數仍然撐不起來（會趨近 0），導致邊界值無法精確控制；
- 太多 knot 還會增加運算成本與過度擬合風險。

✅ 選項二：將邊界節點重複 $k+1$ 次（cubic ⇒ 重複4次）

（也就是使用 augmented knot vector，例如 `augknt`）

➕ 優點（很實用）：

- 可以讓最左與最右的基底函數「貼齊邊界」達到峰值，**從頭到尾都能有效控制 spline 行為**；
- 避免邊界「飄起來」或「掉下去」的問題；
- 是標準作法，數值穩定又容易與 boundary condition 結合（例如強制值或斜率）；
- 用更少的 knot 就能覆蓋整個區間。

➖ 缺點：

- 你無法在中間區域任意調整解析度（但這通常不是大問題）；
- 如果你真的只用很少的 knot，那 spline 在中間也會變鈍（不夠靈活）。

🧠 簡單比喻：

想像你要鋪一條光滑的軌道，讓一輛車從 0 開到 10：

- 若你**只在中間釘很多釘子（加密 knot）**，但頭尾鬆鬆的，軌道在端點就會「浮起來」或「翹起來」；
- 若你**把頭尾多釘幾釘子（重複 knot）**，雖然中間釘得不多，但整條軌道就會很穩、邊界也很牢靠。

CDI method (Conditional density integration)

Use finite order cubic B-splines to approximate the function g.

$$g(y) \approx \sum_{j=1}^{b} \theta_j B_j(y),$$

$$\int_{-\infty}^{X} g(y) d\mathbb{F}^{Q}(y) \approx \sum_{j=1}^{b} \theta_j \int_{-\infty}^{X} B_j(y) d\mathbb{F}^{Q}(y).$$

$$\widehat{g_{t,s}} = \hat{\theta}' B, \tag{C2}$$

where $\theta = (\theta_1, ..., \theta_b)'$ and $\hat{\theta} = (\hat{\theta}_1, ..., \hat{\theta}_b)'$. B is a data matrix where each row is a spline basis function evaluated over the support of our return data.

CDI method (Conditional density integration)

少一個$\frac{1}{T}$

$$\hat{\theta} = \underset{\theta \in \mathbb{R}^b}{\operatorname{argmin}} \sum_{j=1}^{m} \left(\sum_{t=1}^{T} \left(\underbrace{\sum_{i=1}^{b} \theta_i \int_{-\infty}^{X_t} B_i(y) d\mathbb{F}_t^Q(y)}_{\hat{g}(\theta)} \right)^j - \frac{1}{j+1} \right)^2 ,$$

where we use the fact that the jth moment of the uniform distribution over the unit interval is equal to $\frac{1}{j+1}$ and we use the first m moments in estimating the vector θ.

均勻分佈的特性： 均勻分佈 $U[0,1]$ 的每個值出現的概率相等，這意味著它的矩值有簡單的數學形式：

$$E[x^j] = \frac{1}{j+1}.$$

$$\hat{\theta} = \arg\min_{\theta} \sum_{j=1}^{m} \left(\hat{E}[x^j] - \frac{1}{j+1} \right)^2$$

CDI method (Conditional density integration)

$$\widehat{g_{t,s}} = \hat{\theta}' B, \tag{C2}$$

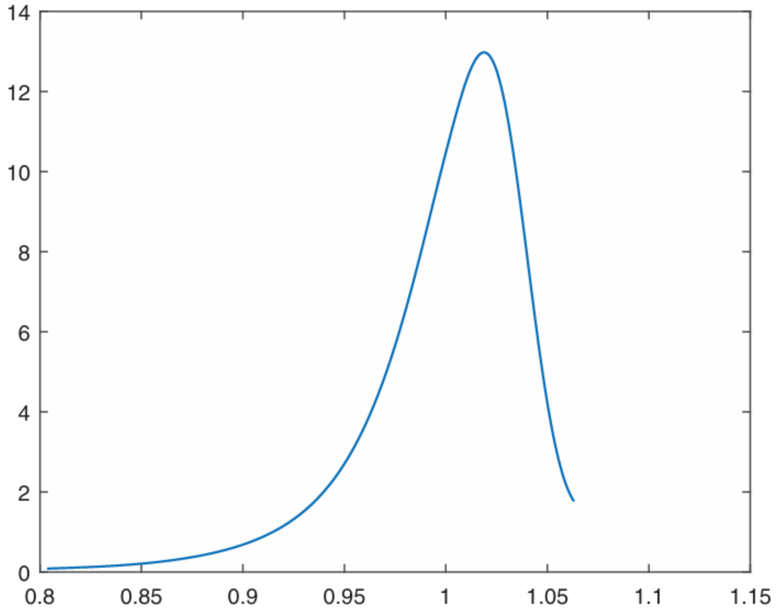
where $\theta = (\theta_1, ..., \theta_b)'$ and $\hat{\theta} = (\hat{\theta}_1, ..., \hat{\theta}_b)'$. B is a data matrix where each row is a spline basis function evaluated over the support of our return data.

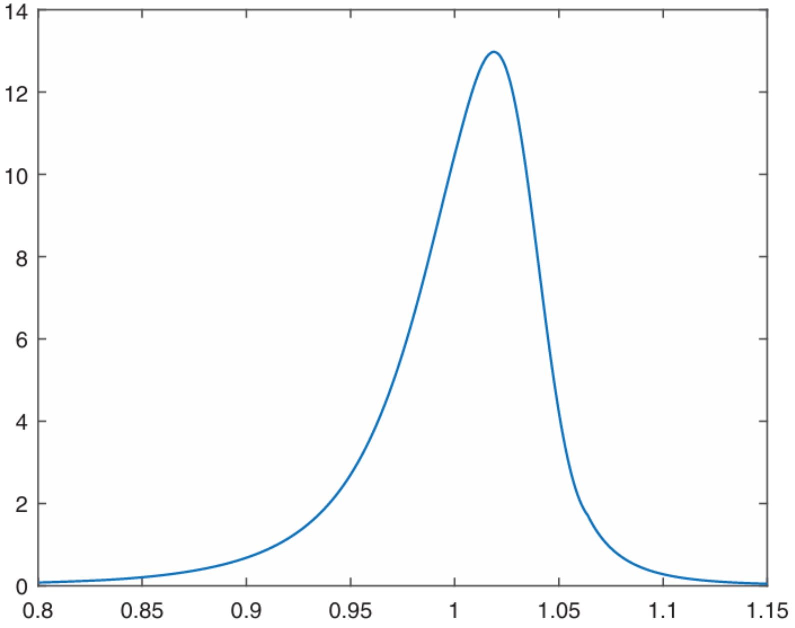
$$\hat{\theta} = \underset{\theta \in \mathbb{R}^b}{\operatorname{argmin}} \sum_{j=1}^{m} \left(\sum_{t=1}^{T} \left(\underbrace{\sum_{i=1}^{b} \theta_i \int_{-\infty}^{X_t} B_i(y) d\mathbb{F}_t^Q(y)}_{\hat{g}(\theta)} \right)^j - \frac{1}{j+1} \right)^2 ,$$

Once we have the estimated $\hat{\theta}$, it is straight forward to estimate g. We simply need to substitute $\hat{\theta}$ into Equation (C2) to obtain our estimate for g, the inverse of the Radon-Nikodym derivative, $\frac{d\mathbb{F}_t^Q}{d\mathbb{F}_t^P}$, for all t. By Equation (2), $\frac{d\mathbb{F}_t^Q}{d\mathbb{F}_t^P} = \frac{1}{\widehat{g_{t,s}}}$ for all t. So our estimated SDF is given by $e^{-r_t \tau} \frac{1}{\widehat{g}(x_{t+s})}$,

Paper Q density

1. 排除best bids (or last prices when bids are not available) < $3/8的選擇權數據，低價往往會提供誤導性的數據，因為它們處於 deep in the money or deep out of the money
2. 使用strike price to creat implied volatility curve
3. convert the implied volatility curve back to the price space (生成更多選擇權價格)
4. price對strike price二次為分計算Q density
5. 1.的數據篩選造成truncated Q density
6. 需使用generalized Pareto distribution補足tail的數據





Our model
1. 訓練時即考慮deep moneyness
2. 可正確預測價格，錯價風險較小
3. 較好的Q density